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Termination of Bergman series. Connection to the B_n Toda system

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Abstract. Truncation of Bergman-type series applied to a canonical hyperbolic equation is shown to lead to the integrable B_n Toda lattice. In particular, the recurrence relations for the truncated Bergman series are implied by the linear representation for the Toda system. To conclude, the action of the Moutard transformation on Bergman series is investigated.

Key words: Bergman series, Toda lattice, Moutard transformation.

1. Introduction

The Bergman integral-operator method was originally developed in connection with the analysis of the hodograph equations in plane, subsonic gasdynamics. The mathematical foundations of the subject are set down in the monograph by Bergman [1]. The physical applications in gasdynamics are described by von Mises and Schiffer [2].

Bergman-series techniques were subsequently applied in the elastostatics of layered materials in order to calculate stress and displacement distributions arising out of a variety of crack problems by Rogers, Clements *et al.* [3–5]. The Bergman-series method was also used to solve contact boundary-value problems in anisotropic thermo-elastostatics in [6]. In each case the application of the procedure to generate analytic solutions involved the truncation of Bergman-series. Bergman-type expansions may also be constructed for classes of hyperbolic problems. In that context, they constitute a variant of the wave-front expansions developed in elastodynamics by Karal and Keller [7] based on work in geometric optics (Luneberg [8]). The method has been used to treat a wide range of initial/boundary-value problems in elastodynamics [9–12], visco-elastodynamics [13, 14] and elasto-plastic boundary propagation [15]. Truncation of Bergman series allows the construction of exact solutions to such dynamical problems.

Here the truncation of an extended Bergman series, originally introduced in [16] in a study of electromagnetic wave propagation through nonlinear dialectric media, is shown to be associated with the integrable B_n Toda lattice system [17]. Indeed, the recurrence relations which determine the truncated Bergman series are implied by the linear representations for the B_n Toda system. In conclusion, it is shown that the classical Moutard transformation maps within the class of Bergman series. In particular, the Moutard transformation maps truncated Bergman series to Bergman series which are truncated at the next level.

2. The Bergman series and its truncation

Here, solutions to the canonical hyperbolic equation

$$\Phi_{rs} = \Lambda(r, s)\Phi \tag{2.1}$$

are sought in terms of a Bergman series of the type [16]

$$\Phi = \sum_{n=0}^{\infty}[\alpha_n(r, s)\Phi_n(r) + \beta_n(r, s)\Psi_n(s)], \tag{2.2}$$

where

$$\Phi_n' = \Phi_{n-1}, \qquad \Psi_n' = \Psi_{n-1}. \tag{2.3}$$

On insertion of (2.3) in (2.2), it is seen that α_n, β_n are determined by the following recurrence relations:

$$\begin{aligned} &\alpha_{n-1,rs} + \alpha_{n,s} - \Lambda\alpha_{n-1} = 0, \\ &\beta_{n-1,rs} + \beta_{n,r} - \Lambda\beta_{n-1} = 0, \end{aligned} \tag{2.4}$$

for $n \geqslant 0$, where, by convention, $\alpha_i = \beta_i = 0$ for $i < 0$. In particular, it follows that

$$\alpha_{0,s} = \beta_{0,r} = 0. \tag{2.5}$$

If the series truncates at order N, that is, if $\alpha_n = \beta_n = 0$ for $n > N$, then this occurs subject to a compatibility condition on Λ which may be expressed in terms of quantities P_n, $(n = 0, 1, \ldots, N-1)$ defined by

$$\begin{aligned} &P_n = P_{n-1} - (\log\ P_0P_1 \ldots P_{n-1})_{rs}, \\ &P_0 = \Lambda. \end{aligned} \tag{2.6}$$

In order to establish this compatibility condition, we will use a relation between the coefficients α_n, β_n in the Bergman series and α_0, β_0 embodied in the following

THEOREM 1. *For* $n \geqslant 0$,

$$\partial_s P_{n-1}^{-1} \ldots \partial_s P_0^{-1} \partial_s(\alpha_{n+1}) = P_n\alpha_0 \tag{2.7}$$

and

$$\partial_r P_{n-1}^{-1} \ldots \partial_r P_0^{-1} \partial_r(\beta_{n+1}) = P_n\beta_0. \tag{2.8}$$

Proof. To establish (2.7), we first note that for $n \geqslant 0$,

$$\begin{aligned} &\partial_s P_n^{-1} \partial_s P_{n-1}^{-1} \ldots \partial_s P_0^{-1} \\ &\quad = [P_{n+1} - (\log P_0P_1 \ldots P_n)_r\partial_s - \partial_r\partial_s]P_n^{-1}\partial_s P_{n-1}^{-1} \ldots \partial_s P_0^{-1}. \end{aligned} \tag{2.9}$$

This result is proven by induction on n. Equation (2.9) holds for $n = 0$ since

$$\partial_s P_0^{-1}(P_0 - \partial_r\partial_s) - [P_1 - (\log\ P_0)_r\partial_s - \partial_r\partial_s]P_0^{-1}\partial_s$$
$$= \partial_s[-P_0^{-1}\partial_r P_0 + (\log P_0)_r + \partial_r]P_0^{-1}\partial_s = 0.$$

Now suppose that (2.9) holds for $n = k \geqslant 0$. Then

$$\partial_s P_{k+1}^{-1}\partial_s P_k^{-1} \ldots \partial_s P_0^{-1}$$
$$= \partial_s P_{k+1}^{-1}[[P_{k+1} - (\log\ P_0P_1 \ldots P_k)_r\partial_s - \partial_r\partial_s]P_k^{-1}\partial_s P_{k-1}^{-1} \ldots \partial_s P_0^{-1}].$$

Hence, (2.9) holds for $n = k + 1$ if

$$\partial_s P_{k+1}^{-1}[P_{k+1} - (\log P_0P_1 \ldots P_k)_r\partial_s - \partial_r\partial_s]$$
$$-[P_{k+2} - (\log\ P_0P_1 \ldots P_{k+1})_r\partial_s - \partial_r\partial_s]P_{k+1}^{-1}\partial_s = 0,$$

and this is readily established.

Then, using (2.4) in the form

$$P_0\alpha_k - \alpha_{k,rs} = \alpha_{k+1,s}, \quad k \geqslant 0$$

we obtain from (2.9) that, for $k \geqslant 1$,

$$\partial_s P_{k-1}^{-1} \ldots \partial_s P_0^{-1}(\alpha_{k+1,s})$$
$$= [P_k - (\log\ P_0P_1 \ldots P_{k-1})_r\partial_s - \partial_r\partial_s]P_{k-1}^{-1}\partial_s P_{k-2}^{-1} \ldots \partial_s P_0^{-1}(\alpha_{k,s}). \tag{2.10}$$

Equation (2.7) then follows by induction on n: for $n = 0$, (2.7) is simply a restatement of the relation $\alpha_{1,s} = P_0\alpha_0$ given by (2.4) when $n = 1$. If (2.7) holds for $n = k - 1 \geqslant 0$ then, from (2.10),

$$\partial_s P_{k-1}^{-1} \ldots \partial_s P_0^{-1}(\alpha_{k+1,s})$$
$$= [P_k - (\log\ P_0P_1 \ldots P_{k-1})_r\partial_s - \partial_r\partial_s]\partial_s P_{k-1}^{-1}(P_{k-1}\alpha_0) = P_k\alpha_0,$$

since $\alpha_{0,s} = 0$, and so then (2.7) holds for $n = k$.

The second part of the theorem is proved by changing α to β and s to r *mutatis mutandis*.□

Accordingly, we obtain the truncation criterion as originally set down in [18] from Theorem 1 for $n = N$.

COROLLARY 2. *The Bergman series* (2.2) *may be terminated at order N (so that α_n, β_n may be taken to be zero for $n > N$) if Λ satisfies the condition $P_N = 0$, that is*

$$P_{N-1} - (\log\ P_0P_1 \ldots P_{N-1})_{rs} = 0, \tag{2.11}$$

where $P_0 = \Lambda$ and $P_n = P_{n-1} - (\log\ P_0P_1 \ldots P_{n-1})_{rs}$ for $n = 1, \ldots, N - 1$.

In particular, in the first nontrivial case $N = 1$, so that $\alpha_n = \beta_n = 0$, $n = 2, 3, \ldots$ it is seen that the conditions (2.11) reduce to Liouville's equation

$$\theta_{rs} = \mathrm{e}^{\theta} \tag{2.12}$$

in $\theta = \log \Lambda$.

The following properties of the Bergman series coefficients also follow immediately from (2.5) and Theorem 1.

COROLLARY 3. *For all $k \geqslant 0$,*

$$\partial_s P_{k-1}^{-1} \dots \partial_s P_0^{-1} \partial_s(\alpha_k) = \partial_r P_{k-1}^{-1} \dots \partial_r P_0^{-1} \partial_r(\beta_k) = 0. \tag{2.13}$$

3. Connection to the B_n Toda system

There is an integrable two-dimensional Toda system associated with each semi-simple and affine Lie algebra, namely

$$\theta_{n,xt} = -\sum_m M_{n,m}\, \mathrm{e}^{-\theta_m}, \tag{3.1}$$

where M is the Cartan matrix of the corresponding Lie algebra. In particular, for the semi-simple Lie algebra B_n $(n > 1)$, the Toda system becomes

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{n-1} \end{bmatrix}_{xt} = -\begin{bmatrix} 2 & -1 & & & & \\ -2 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{-\theta_0} \\ \mathrm{e}^{-\theta_1} \\ \mathrm{e}^{-\theta_2} \\ \vdots \\ \mathrm{e}^{-\theta_{n-1}} \end{bmatrix}. \tag{3.2}$$

Note that for $n = 1$, corresponding to the Lie algebra $B_1 = A_1$, the system reduces to the Liouville equation (2.12) where $\theta_0 = -\theta + \log 2$, associated with the truncation of the Bergman series at $N = 1$.

Next consider the case when the Bergman series truncates at $N = 2$. The compatibility condition (2.11) is

$$P_1 = (\log\ P_0 P_1)_{rs}, \tag{3.3}$$

where

$$P_1 = P_0 - (\log\ P_0)_{rs}. \tag{3.4}$$

If we set $P_0 = 2\,\mathrm{e}^{-\theta_0}$ and $P_1 = \mathrm{e}^{-\theta_1}$ in (3.3) and (3.4) then we obtain

$$\mathrm{e}^{-\theta_1} = -(\theta_0 + \theta_1)_{rs} \tag{3.5}$$

$$\mathrm{e}^{-\theta_1} = 2\,\mathrm{e}^{-\theta_0} + \theta_{0_{rs}}, \tag{3.6}$$

namely, the B_2 Toda system

$$\begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}_{rs} = -\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{-\theta_0} \\ \mathrm{e}^{-\theta_1} \end{bmatrix}. \tag{3.7}$$

In general, if we set

$$P_0 = 2\,\mathrm{e}^{-\theta_0}, \qquad P_k = \mathrm{e}^{-\theta_k}, \qquad k = 1, \dots, N-1 \tag{3.8}$$

then the criterion for truncation of the Bergman series at order N (2.11) together with the definitions (2.6) deliver the B_N Toda system, (3.2) with $N = n$.

4. The A_∞ Toda-system setting

The B_n Toda system may be approached as a reduction of the general A_∞ Toda system, (3.1) with

$$M = \begin{bmatrix} \ddots & \ddots & \ddots & & & & & \\ & -1 & 2 & -1 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & -1 & 2 & -1 & & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4.1}$$

The linear system

$$\begin{aligned} \phi_{n,x} &= v_{n,x}\phi_n + \phi_{n-1}, \\ \phi_{n,t} &= \mathrm{e}^{-\theta_n}\phi_{n+1}, \end{aligned} \qquad n \in \mathbb{Z} \tag{4.2}$$

where $\theta_n = v_{n+1} - v_n$ is the Lax pair for (3.1) with M given by (4.1). That is, the nonlinear system (3.1) is the compatibility condition for the linear system (4.2). Darboux and binary Darboux transformations are known for this system [19] and the particular reductions [20] discussed below.

The B_∞ reduction is obtained via the specialization $v_{-n} = -v_n$ for all n. This means in particular that $v_0 = 0$ and $\theta_{-n} = \theta_{n-1}$ whence

$$\theta_{-1,xt} = \theta_{0,xt} = -(\mathrm{e}^{-\theta_0} - \mathrm{e}^{-\theta_1}) \tag{4.3}$$

$$\theta_{-n-1,xt} = \theta_{n,xt} = -(-\mathrm{e}^{-\theta_{n-1}} + 2\,\mathrm{e}^{-\theta_n} - \mathrm{e}^{-\theta_{n+1}}), \quad n \geqslant 1. \tag{4.4}$$

Redefining $\theta_0 \to \theta_0 - \log 2$, we obtain the B_∞ Toda system (3.1) with

$$M = \begin{bmatrix} 2 & -1 & & & & & & \\ -2 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & -1 & 2 & -1 & & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4.5}$$

Although the number of independent variables is halved in this B_∞ reduction, the linear representation involves a full complement of components ϕ_n, where $n \in \mathbb{Z}$. This reduced linear system has, however, a self-contained semi-infinite subsystem involving only ϕ_n, where $n \geqslant 0$,

$$\begin{aligned} &\phi_{0,xt} = 2\,\mathrm{e}^{-\theta_0}\phi_0, \qquad \phi_{0,t} = 2\,\mathrm{e}^{-\theta_0}\phi_1, \\ &\phi_{k,x} = v_{k,x}\phi_k + \phi_{k-1}, \qquad \phi_{k,t} = \mathrm{e}^{-\theta_k}\phi_{k+1}, \quad (k \geqslant 1) \end{aligned} \tag{4.6}$$

which still has the B_∞ Toda system as its compatibility condition. Moreover, all ϕ_k may be obtained recursively, without further quadrature, in terms of ϕ_0:

$$\phi_1 = \tfrac{1}{2}\,\mathrm{e}^{\theta_0}\phi_{0,t}, \tag{4.7}$$

$$\phi_{k+1} = \mathrm{e}^{\theta_k}\phi_{k,t}, \quad (k \geqslant 1). \tag{4.8}$$

The further reduction to B_n is obtained by a limiting process from B_∞. For all $k > 0$, take v_{n+k} to be constant (with respect to x and t) and ordered such that $\theta_{n+k} = v_{n+k+1} - v_{n+k} > 0$. Then let the constants $v_{n+k} \to \infty$ while maintaining this ordering and so that $\theta_{n+k} \to \infty$. Note that $\theta_n = v_{n+1} - v_n \to \infty$ also. The resulting nonlinear system is the B_n Toda system (3.2) while the reduced linear system, giving a Lax pair for (3.2), is

$$\begin{aligned} &\phi_{0,xt} = 2\,\mathrm{e}^{-\theta_0}\phi_0, \qquad \phi_{0,t} = 2\,\mathrm{e}^{-\theta_0}\phi_1, \\ &\phi_{k,x} = v_{k,x}\phi_k + \phi_{k-1}, \qquad (k = 1, \ldots, n), \\ &\phi_{k,t} = \mathrm{e}^{-\theta_k}\phi_{k+1}, \qquad (k = 1, \ldots, n-1), \quad \phi_{n,t} = 0. \end{aligned} \tag{4.9}$$

In the next section, we will show that this linear representation for the B_n Toda system may be used to determine the terms in the truncated Bergman series.

5. Lax pair of B_n and Bergman series

The coefficients α_k, β_k in the Bergman series and the components ϕ_k in the linear representation of B_n are related. However, the connection between them seems nontrivial.

To determine the coefficients α_k in terms of ϕ_k we first identify the independent variables $x = r, t = s$. We will illustrate this process for small values of the truncation order N. We find relationship between β_k and ϕ_k by the same process using the alternative identification $x = s, t = r$.

The coefficients α_k in the Bergman series truncated at order 1 are determined by

$$\alpha_{0,s} = 0, \qquad \alpha_{1,s} = P_0\alpha_0, \qquad 0 = P_0\alpha_1 - \alpha_{1,rs}, \tag{5.1}$$

while from the linear representation of $A_1 (= B_1)$ we have

$$\phi_{0,xt} = P_0\phi_0, \qquad \phi_{0,t} = P_0\phi_1, \qquad \phi_{1,t} = 0. \tag{5.2}$$

Recall that the coefficient Λ in the hyperbolic Equation (2.1) is written as $P_0 = 2\,\mathrm{e}^{-\theta_0}$ and $P_k = \mathrm{e}^{-\theta_k}$ for $k > 0$. Clearly we may choose $\alpha_0 = \phi_1$ and $\alpha_1 = \phi_0$. Similarly, for $N = 2$,

$$\alpha_{0,s} = 0, \qquad \alpha_{1,s} = P_0\alpha_0, \qquad \alpha_{2,s} = P_0\alpha_1 - \alpha_{1,rs}, \qquad 0 = P_0\alpha_2 - \alpha_{2,rs}, \tag{5.3}$$

while from the linear representation of B_2 we have

$$\phi_{0,xt} = P_0\phi_0, \qquad \phi_{0,t} = P_0\phi_1, \qquad \phi_{1,t} = P_1\phi_2, \qquad \phi_{2,t} = 0. \tag{5.4}$$

Taking an *ansatz* of the form

$$\alpha_0 = \phi_2, \qquad \alpha_1 = a\phi_1 + A(x,t)\phi_2, \quad \alpha_2 = b\phi_0, \tag{5.5}$$

were a, b are constant we find that $a = b = 1$ and $A = -\theta_{1,x}$.

For any N one may show that a similar *ansatz* will lead to expressions for the coefficients α_k in terms of ϕ_k.

6. Bergman series and the Moutard transformation

Here, we investigate the action of the classical Moutard transformation [21] on (semi-)infinite and truncated Bergman series. The Moutard transformation plays an important role in Soliton Theory [22–24] and may be formulated as follows:

THEOREM 4. *If Φ and ρ are two solutions of the hyperbolic equation*

$$\Phi_{rs} = \Lambda\Phi \tag{6.1}$$

and the corresponding bilinear potential S is defined by

$$S_r = \rho\Phi_r - \rho_r\Phi, \qquad S_s = \rho_s\Phi - \rho\Phi_s, \tag{6.2}$$

then (6.1) *is form-invariant under*

$$\Phi \to \tilde{\Phi} = \frac{S}{\rho}, \qquad \Lambda \to \tilde{\Lambda} = \Lambda - 2(\log\,\rho)_{rs}. \tag{6.3}$$

The problem may be best approached by considering the infinite Bergman series

$$\Phi = \sum_{n=-\infty}^{\infty} [\alpha_n\Phi_n(r) + \beta_n\Psi_n(s)], \tag{6.4}$$

where the functions Φ_n and Ψ_n obey the usual relations

$$\Phi_n' = \Phi_{n-1}, \qquad \Psi_n' = \Psi_{n-1}. \tag{6.5}$$

If we demand that the coefficients α_n, β_n be independent of the functions Φ_n, Ψ_n and the Moutard equation (6.1) hold for any choice of the latter, then the recurrence relations (2.4), that is

$$\alpha_{n,rs} + \alpha_{n+1,s} = \Lambda\alpha_n, \qquad \beta_{n,rs} + \beta_{n+1,r} = \Lambda\beta_n, \tag{6.6}$$

apply. These imply the existence of an infinite number of potentials X_n, Y_n defined according to

$$X_{n,r} = \rho\alpha_{n,r} - \rho_r\alpha_n + \rho\alpha_{n+1} - X_{n+1}, \tag{6.7}$$

$$X_{n,s} = \rho_s \alpha_n - \rho \alpha_{n,s}, \tag{6.8}$$

$$Y_{n,r} = \rho \beta_{n,r} - \rho_r \beta_n, \tag{6.9}$$

$$Y_{n,s} = \rho_s \beta_n - \rho \beta_{n,s} - \rho \beta_{n+1} - Y_{n+1}. \tag{6.10}$$

It is readily verified that the Frobenius system (6.7–6.10) is compatible modulo (6.6).

Now, the crucial observation is that the quantity

$$S = \sum_{n=-\infty}^{\infty} [X_n \Phi_n + Y_n \Psi_n] \tag{6.11}$$

satisfies the defining relations (6.2). Hence, the Moutard transform

$$\tilde{\Phi} = \sum_{n=-\infty}^{\infty} \left[\frac{X_n}{\rho} \Phi_n + \frac{Y_n}{\rho} \Psi_n \right] \tag{6.12}$$

constitutes another infinite Bergman series with potential $\tilde{\Lambda}$ as given by (6.3). We have therefore established that the Moutard transformation with a particular choice of the integration constant in S maps within the class of infinite Bergman series.

In the 'semi-infinite' case represented by

$$\alpha_n = \beta_n = 0, \quad n < 0, \tag{6.13}$$

we set

$$X_n = Y_n = 0, \quad n < 0 \tag{6.14}$$

and regard the relations (6.7), (6.10) as recursive definitions of the remaining potentials, that is

$$\begin{aligned} X_{n+1} &= \rho \alpha_{n,r} - \rho_r \alpha_n + \rho \alpha_{n+1} - X_{n,r}, \quad X_0 = \rho \alpha_0, \\ Y_{n+1} &= \rho_s \beta_n - \rho \beta_{n,s} - \rho \beta_{n+1} - Y_{n,s}, \quad Y_0 = -\rho \beta_0. \end{aligned} \tag{6.15}$$

The relations (6.8) and (6.9) are satisfied identically as can easily be shown by induction on n. Thus, the Moutard transform becomes

$$\tilde{\Phi} = \sum_{n=0}^{\infty} \left[\frac{X_n}{\rho} \Phi_n + \frac{Y_n}{\rho} \Psi_n \right], \tag{6.16}$$

which again is of the form of a semi-infinite Bergman series.

If we now truncate the seed Bergman series at the Nth level, that is

$$\alpha_n = \beta_n = 0, \quad n > N, \tag{6.17}$$

then the definitions (6.15) apparently indicate that, in the generic case, the Moutard transform (6.16) is not finite. However, with respect to a different set of functions $\{\tilde{\Phi}_n, \tilde{\Psi}_n\}$ to be introduced below, the series (6.16) does indeed truncate. This is shown as follows.

In the finite case, the recurrence relations (6.7–6.10) imply that

$$\begin{aligned} X_{N+1+k} &= (-1)^k \partial_r^k X_{N+1}, \qquad X_{N+1} = X_{N+1}(r), \\ Y_{N+1+k} &= (-1)^k \partial_s^k Y_{N+1}, \qquad Y_{N+1} = Y_{N+1}(s), \end{aligned} \qquad k \geqslant 0 \tag{6.18}$$

which suggests introducing the functions

$$\begin{aligned} \tilde{\Phi}_{N+1}(r) &= \sum_{n=N+1}^{\infty} X_n \Phi_n, \qquad \tilde{\Phi}_{N+1-k}(r) = \partial_r^k \tilde{\Phi}_{N+1} \\ \tilde{\Psi}_{N+1}(s) &= \sum_{n=N+1}^{\infty} Y_n \Psi_n, \qquad \tilde{\Psi}_{N+1-k}(s) = \partial_s^k \tilde{\Psi}_{N+1}. \end{aligned} \tag{6.19}$$

Differentiation of $\tilde{\Phi}_{N+1}$, $\tilde{\Psi}_{N+1}$ and simplification by means of (6.18) yields

$$\tilde{\Phi}_N = X_{N+1} \Phi_N, \qquad \tilde{\Psi}_N = Y_{N+1} \Psi_N, \tag{6.20}$$

which shows that the functions $\{\Phi_n, \Psi_n, n \leqslant N\}$ are finite linear combinations of the functions $\{\tilde{\Phi}_n, \tilde{\Psi}_n, n \leqslant N\}$ and *vice versa*. Thus, the series (6.16) assumes the form

$$\tilde{\Phi} = \sum_{n=0}^{N+1} \left[\frac{\tilde{X}_n}{\rho} \tilde{\Phi}_n + \frac{\tilde{Y}_n}{\rho} \tilde{\Psi}_n \right], \tag{6.21}$$

which constitutes a Bergman series truncated at the level $N + 1$ since $\tilde{X}_n$, $\tilde{Y}_n$ and ρ are independent of $\tilde{\Phi}_n$, $\tilde{\Psi}_n$. More precisely, if we insert (6.21) into the tilda version of the Moutard equation (6.1), we obtain an equation of the form

$$\sum_{n=-1}^{N+1} [\gamma_n \tilde{\Phi}_n + \delta_n \tilde{\Psi}_n] = 0. \tag{6.22}$$

This may be re-written as a series in the functions Φ_n and Ψ_n and hence the coefficients in this series have to vanish identically. The latter immediately implies that $\gamma_n = \delta_n = 0$. Accordingly, the coefficients

$$\tilde{\alpha}_n = \frac{\tilde{X}_n}{\rho}, \qquad \tilde{\beta}_n = \frac{\tilde{Y}_n}{\rho} \tag{6.23}$$

satisfy the tilda version of the recurrence relations $(2.4\text{–}2.5)_{N \to N+1}$.

We conclude with some implications of the above analysis. Firstly, the relations (6.20) indicate that the transition from the set $\{\Phi_n\}$ to the set $\{\tilde{\Phi}_n\}$ may be regarded as a 'gauge transformation'. In fact, gauge transformations generically convert truncated Bergman series into semi-infinite Bergman series. Thus, given a 'gauge function' $f = f(r)$ and some integer N, let us introduce the transform

$$\tilde{\Phi}_N = f^{-1} \Phi_N \tag{6.24}$$

together with its hierarchy defined by

$$\tilde{\Phi}_n' = \tilde{\Phi}_{n-1}. \tag{6.25}$$

As pointed out earlier, the functions $\Phi_n, n \leqslant N$ are finite linear combinations of the form

$$\Phi_{N-k} = \mu_{k0}\tilde{\Phi}_{N-k} + \cdots + \mu_{kk}\tilde{\Phi}_N, \quad k \geqslant 0. \tag{6.26}$$

However, the expression

$$\Phi_{N+1} = \int f\tilde{\Phi}_N \, \mathrm{d}r \tag{6.27}$$

may formally be integrated by parts to obtain the infinite expansion

$$\Phi_{N+1} = \sum_{k=0}^{\infty}(-1)^k(\partial_r^k f)\tilde{\Phi}_{N+k+1}. \tag{6.28}$$

Hence, any Bergman series which is truncated above the Nth order is transformed into a semi-infinite Bergman series under the gauge transformation (6.24).

Secondly, since, by definition, $\tilde{X}_{N+1} = \tilde{Y}_{N+1} = 1$, the coefficients $\tilde{\alpha}_{N+1} = \tilde{\beta}_{N+1} = 1/\rho$ satisfy

$$\left(\frac{1}{\rho}\right)_{rs} = \tilde{\Lambda}\left(\frac{1}{\rho}\right). \tag{6.29}$$

Thus, we recover the known fact that the inverse of the eigenfunction that generates the Moutard transformation constitutes another eigenfunction associated with the transformed potential. Interestingly, similar involutory conditions of the form *eigenfunction* $\to$ I (*eigenfunction*), $I^2 = id$ may be used to derive Bäcklund transformations for large classes of integrable equations, in particular the AKNS (Ablowitz–Kaup–Newell–Segur) system [25].

Thirdly, since the Moutard transformation maps truncated Bergman series to Bergman series which are truncated at the next level, the Moutard transformation constitutes a Bäcklund transformation from the B_n Toda lattice to the B_{n+1} Toda lattice. In fact, n-fold application of the Moutard transformation to the seed solution $\Lambda = 0$ leads to the general solution of the B_n Toda lattice. Thus, if $\Lambda = 0$, the general solution of $\rho_{rs} = 0$ is $\rho = f(r) + g(s)$ and the new potential $\tilde{\Lambda}$ reads

$$\tilde{\Lambda} = 2\frac{f_r g_s}{(f+g)^2} \tag{6.30}$$

which represents the general solution of the Liouville equation (2.12). Furthermore, the Moutard transformation may be iterated in a purely algebraic manner [22]. In the present context, this means that after n iterations of the Moutard transformation, the new potential $\Lambda^{(n)}$ is given in terms of n linearly independent solutions of the base equation $\rho_{rs} = 0$. In other words, $\Lambda^{(n)}$ depends on n arbitrary functions of r and n arbitrary functions of s which coincides with the solution space of the B_n Toda lattice.

7. Conclusion

It has been seen how the criterion for truncation of Bergman-series solutions at level N for the canonical hyperbolic Equation (2.1) is associated with the B_N Toda-lattice system. This connection with an integrable system led naturally to a study of the action of the classical Moutard transformation on truncated Bergman series. A Bäcklund transformation mapping Bergman series truncated at level N to Bergman series truncated at level $N+1$ was thereby established. These results have important physical applications in continuum mechanics. In particular, Bergman-series theory has been applied by Rogers *et al.* [16] to the hyperbolic Equation (2.1) in the analysis of the one-dimensional polarised electromagnetic waves in media characterised by nonlinear constitutive relations $\boldsymbol{D} = \boldsymbol{D}(|\boldsymbol{E}|)$, $\boldsymbol{B} = \boldsymbol{B}(|\boldsymbol{H}|)$ where $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ denote, in turn, the electric displacement field, the magnetic flux, the electric and magnetic field. Truncation of Bergman series at level $N = 1$ was discussed therein and the associated multi-parameter constitutive laws which allow such termination were constructed. The results presented here realised via established methods of soliton theory now allow the iterative construction of such Bergman series with truncation at arbitrary level. The concomitant nonlinear constitutive laws corresponding to such termination may be used to model real physical behaviour in the manner described by Kazakia and Venkataraman [26]. These procedures are readily adapted to the analysis of one-dimensional pulse propagation in nonlinear elastic media as treated extensively via the model constitutive law approach by Cekirge and Varley [27] and Kazakia and Varley [28, 29].

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